Guarantees for Spectral Clustering with Fairness Constraints
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Spectral clustering (SC)

SC is the method of choice for clustering the nodes of a graph. Example: to find clusters in a friendship network — many connections within clusters, few connections between clusters.

There are several versions of SC (von Luxburg, 2007), here we focus on unnormalized SC but our results hold similarly for normalized SC too.

Formally:

\[ G = (V, E) \] is an undirected graph on \[ V = [n] \]. \( W \in \mathbb{R}^{n \times n} \) is a (weighted) adjacency matrix with \( W_{ii} > 0 \) if there is an edge between \( i \) and \( j \). Let \( k \) be the number of clusters. SC aims to partition \( V \) into \( k \) clusters with minimum value of the RatioCut objective function defined as follows: for a clustering \( V = C_1 \cup \ldots \cup C_k \) it is

\[
\text{RatioCut}(C_1, \ldots, C_k) = \sum_{i=1}^{k} \frac{\text{Cut}(C_i, V \setminus C_i)}{|C_i|} = \sum_{i=1}^{k} \sum_{j \notin C_i} W_{ij}
\]

Let \( D \) be the degree matrix (with \( D_{ii} = \sum_{j \in V} W_{ij} \)) and \( L = D - W \) be the graph Laplacian matrix.

Encoding a clustering \( V = C_1 \cup \ldots \cup C_k \) by \( H \in \mathbb{R}^{n \times k} \) with

\[
H_{ij} = \begin{cases} \sqrt{D_{ii}} & i \in C_j, \\ 0 & i \notin C_j \end{cases}
\]

we have \( \text{RatioCut}(C_1, \ldots, C_k) = \text{Tr}(H^T L H) \). SC relaxes the exact problem

\[
\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T L H) \quad \text{subject to } H^T H = I_k
\]

and solves

\[
\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T L H) \quad \text{subject to } H^T H = I_k
\]

instead. A solution to (3) is given by a matrix \( H \) that contains some orthonormal eigenvectors corresponding to the \( k \) smallest eigenvalues of \( L \). Then, a clustering of \( V \) is inferred from \( H \) by applying \( k \)-means clustering to the rows of \( H \).

Spectral clustering with fairness constraints

Let \( V \) contain \( h \) demographic groups \( V_s \) such that \( V = \cup_s V_s \). Chierichetti et al. (2017) proposed a notion of fairness for clustering: in every cluster, each group \( V_s \) should be represented with (approximately) the same fraction as in the whole data set \( V \).

We can incorporate this goal into (2) via a linear constraint on \( H \):

\[
\text{Analysis on variant of the stochastic block model (SBM)}

We analyze our fair version of SC on a natural variant of the famous SBM (Holland et al., 1983) where \( V = [n] \) comprises \( h \) groups \( V = \cup_s V_s \) and partitioned into \( k \) ground-truth clusters \( V = C_1 \cup \ldots \cup C_k \) such that \( (V_s \cap C_l)/|C_l| = \eta_s \), \( s \in [h], l \in [k] \), for some \( \eta_s \in (0, 1) \). This ground-truth clustering is perfectly fair. We define a random graph on \( V \) by connecting vertices \( i \) and \( j \) with probability \( P_{ij} \), where

\[
Pr(i, j) = \begin{cases} a, & i \text{ and } j \text{ in same cluster and in same group,} \\ b, & c, & i \text{ and } j \text{ not in same cluster, but in same group,} \\ c, & d, & i \text{ and } j \text{ not in same cluster and not in same group,} \\ d, & \end{cases}
\]

for some \( a > b > c > d \). An example of such a graph can be seen in the next figure.

For such a graph, standard SC is likely to return the unfair clustering \( V = V_1 \cup V_2 \) while Algorithm 1 returns the fair clustering \( V = C_1 \cup C_2 \).

Example of a graph generated from our variant of the SBM. There are two meaningful clusters into two clusters: \( V = C_1 \cup C_2 \) and \( V = V_1 \cup V_2 \). However only the first one is fair.

Experiments

SBM

Error (fraction of “misclassified” vertices) as a function of \( n \). In the third plot, \( C_1 \), \( C_2 \) and \( C_3 \) have different sizes and Assumption (6) is not satisfied.

Real networks

Average balance of clusters and RatioCut value as a function of \( k \).

References


